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by

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ASYMPTOTIC THEORY FOR NONPARAMETRIC CONFIDENCE INTERVALS

Peter W. Glynn

1. Introduction

The problem of assigning nonparametric confidence intervals has recently been the focus of renewed attention. One impetus has been the development of the "bootstrap" method by EFRON (1979) as a general nonparametric statistical tool. BICKEL and FREEDMAN (1981), as well as SINGH (1981), have shown that the bootstrap's distributional approximation is asymptotically valid in a wide variety of circumstances, while EFRON (1981) has studied, in particular, the bootstrap's viability for setting confidence intervals. The recognition that computing power is increasingly available has allowed statisticians to consider confidence interval methods, such as the bootstrap, that are computationally more complex but statistically better behaved than previous techniques. The pivotal transformation of JOHNSON (1978) is another such procedure.

Nonparametric confidence interval methodology has also attracted considerable study in the Monte Carlo simulation literature; see CRANE and LEMOINE (1977), FISHMAN (1978), and LAW and KELTON (1982), for example. The idea is to assign confidence intervals to point estimators obtained from a simulation output sequence, in order to give the simulator an assessment of the estimates' variability.

The simulation applications mentioned above dictate that we analyze the confidence interval problem for ratio estimators. To be

precise, we shall consider the problem of estimating $r = EY_n/E\tau_n$ from a sequence of independent and identically distributed (i.i.d.) random vectors (r.v.'s) $\{(Y_n, \tau_n);\ n \geq 1\}$, where $E(|Y_n| + |\tau_n|) < \infty$ and $E\tau_n \neq 0$. Of course, the classical nonparametric situation is captured as a special case, by setting $\tau_n \equiv 1$.

The organization of this chapter is as follows. In Section 2, we show that ratio estimators arise naturally in the context of the simulation and/or statistical analysis of ergodic quantities associated with regenerative stochastic processes. Section 3 discusses the basic central limit theorem (CLT) on which all the confidence interval methods to be considered in this chapter will be based. Asymptotic error analysis of these techniques requires certain tools from the theory of Edgeworth expansions. In Section 4, results of BHATTACHARAYA and GHOSH (1978) and GOTZE and HIPP (1978) are extended to accomodate the generalizations required by the ratio estimator problem.

In Section 5, we obtain a rigorous Edgeworth expansion for the ratio estimator pivot statistic. This extends the work of CHUNG (1946) from the classical case to the ratio problem (the formulas there contain some errors, however; see WALLACE (1958), p. 642). This enables us, in Section 6, to analyze the error asymptotics of the ratio pivot confidence interval, as well as two related intervals. In particular, we are able to precisely identify the effect of the Student t-correction (i.e., using Student t-quantiles rather than normal quantiles in the limit approximation) on coverage.

In Section 7, we extend Johnson's pivotal transformation to ratio pivots, and show that it corrects for the asymmetry effects of order $n^{-1/2}$ (n is the sample size) that occur in the standard pivot. Section 8 presents a second-order pivotal transformation which corrects coverage error in the standard pivot to order n^{-1}. It turns out that this second-order pivot is the nonparametric analogue of a transformation suggested by HOTELLING and FRANKEL (1938) to "normalize" Student t-variates. Section 9 discusses computational issues and displays results of Monte Carlo sampling experiments in which the coverage characteristics of the pivotal transformations were compared with those of the untransformed pivot.

2. Some Applications of Ratio Estimator Confidence Intervals

The possibility of extending confidence interval methodology from the classical framework to the ratio estimator context has been previously studied in the statistical literature. For example, ROY and POTTHOFF (1958) discuss this problem in the case where (Y_n, τ_n) has a bivariate normal distribution. Their motivation stemmed from applications in which a comparison of EY_n and $E\tau_n$, in terms of their ratio, is desired. For instance, in evaluating the effect of a treatment, the ratio of the mean of the treated population to the mean of the untreated population is of interest.

More recently, this problem has attracted considerable attention in the simulation community. Consider a measurable regenerative stochastic process $\{X_t;\ t \geq 0\}$ (see SMITH (1955) for a complete

discussion). Then, there exist random times $T_1 < T_2 < \cdots$ with $T_n \to \infty$ such that the vectors $\{(Y_k(f), \tau_k);\ k \geq 1\}$ are i.i.d., where

$$Y_k(f) = \int_{T_k}^{T_{k+1}} f(X_s)ds$$

$$\tau_k = T_{k+1} - T_k$$

for any suitably measurable real-valued function f. It can be shown (see [13]) that if $E(Y_n(|f|) + \tau_n) < \infty$, then

$$\int_0^t f(X_s)ds/t \to r(f) \equiv EY_n(f)/E\tau_n \quad \text{a.s.}$$

Hence, development of confidence intervals for ergodic quantities such as $r(f)$, in the context of regenerative processes, leads naturally to the study of ratio estimators. For a complete discussion of the simulation issues related to regenerative ratio estimates, we refer the reader to IGLEHART (1978), and Chapter 6 of RUBINSTEIN (1981).

Of course, it is clear that the regenerative approach is equally applicable to the nonparametric analysis of statistical data modelled as a regenerative stochastic process (for example, finite state Markov chains).

3. Confidence Intervals for Ratio Estimators

For the remainder of this chapter, we assume that $E(|Y_n|+|\tau_n|)<\infty$, $E\tau_n \neq 0$, and $0 < \sigma^2(Z_1) < \infty$, where $Z_k = Y_k - r\tau_k$. Also, without loss of generality, we assume that $E\tau_n > 0$ (otherwise, we pass to $(-Y_n, -\tau_n)$). For a generic sequence $\{\eta_i;\ i \geq 1\}$ of i.i.d. r.v's, we shall use the notation $\bar{\eta}_k = \sum_{i=1}^k \eta_i/k$, and $\hat{\eta}_k = k^{1/2}(\bar{\eta}_k - E\eta_1)$. The r.v.'s

$$r_n = I_{\{\bar{\tau}_n \neq 0\}} \bar{Y}_n/\bar{\tau}_n$$

$$v_n = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - r_n\tau_i)^2$$

$$t_n = I_{\{v_n>0,\bar{\tau}_n\neq 0\}}\, n^{1/2} \frac{(r_n - r)}{v_n^{1/2}}\, \bar{\tau}_n$$

(we interpret a product involving an indicator to be zero if the indicator is zero) play an important role in ratio estimator confidence intervals. To be precise, it is not difficult to show that $r_n \to r$ a.s. and that

$$F_n(x) \equiv P\{t_n \leq x\} \to \int_{-\infty}^{x} \phi(u)du \equiv \Phi(x) \tag{3.1}$$

where $\phi(u) = (2\pi)^{-1/2} \exp(-u^2/2)$. The CLT (3.1) proves that $[L_n(p), R_n(p)]$ $(0 \le p \le \alpha)$ is an approximate 100 $(1-\alpha)$% confidence interval for r, where

$$L_n(p) = r_n - \nu_n(p+1-\alpha)\, I_{\{\bar\tau_n>0\}} - \nu_n(p) I_{\{\bar\tau_n<0\}}$$

$$R_n(p) = r_n - \nu_n(p)\, I_{\{\bar\tau_n>0\}} - \nu_n(p+1-\alpha)\, I_{\{\bar\tau_n<0\}} \tag{3.2}$$

$$\nu_n(p) = z(p)\, v_n^{1/2}/(n^{1/2}\, \bar\tau_n)$$

and $z(p) = \Phi^{-1}(p)$. In order to study the error asymptotics of the above intervals, we introduce the error descriptors

$$\varepsilon_n^{\ell}(p) = P\{r < L_n(p)\} - (\alpha-p)$$

$$\varepsilon_n^{r}(p) = P\{r \ge R_n(p)\} - p \tag{3.3}$$

$$\varepsilon_n(p) = P\{L_n(p) \le r < R_n(p)\} - (1-\alpha)\ .$$

The term $\varepsilon_n(p)$ measures the coverage probability error in the interval $[L_n(p), R_n(p)]$, whereas the terms $\varepsilon_n^{\ell}(p)$, $\varepsilon_n^{r}(p)$ provide the one-sided coverage errors. The one-sided errors will assist us in evaluating the degree to which the above nonparametric confidence interval captures the asymmetry which is present in parametric confidence intervals (see [9] for a discussion of this point).

In analogy with the classical nonparametric case, two other intervals are natural to consider. Let t_n^* be the pivot obtained from t_n by replacing v_n with v_n^*, where $v_n^* = ((n-1)/n)v_n$. Intervals $[L_n^*(p), R_n^*(p)]$, with errors $\varepsilon_n^{\ell}(p)^*$, $\varepsilon_n^{r}(p)^*$, and $\varepsilon_n(p)^*$, are defined analogously to (3.2) and (3.3).

A second alternative is to use the Student t-distribution with k degrees of freedom. Let $z_k(p)$ be the p'th quantile of such a distribution. Then, $z_k(p) \to z(p)$ as $k \to \infty$ (PEISER (1943)), and thus, in light of the parametric theory for the bivariate normal case, it is of interest to consider the intervals, $[L_n'(p), R_n'(p)]$, with errors $\varepsilon_n^{\ell}(p)'$, $\varepsilon_n^{r}(p)'$, and $\varepsilon_n(p)'$, constructed by substituting $z_{n-1}(p)$ for $z(p)$ in $\nu_n(p)$.

Before concluding this section, we observe that if $E|\tau_n|^{s+2} < \infty$ for $s \geq 0$, Chebyshev's inequality implies that

$$P\{\bar{\tau}_n \leq 0\} \leq E(\hat{\tau}_n)^{\kappa}/(n^{1/2}\, E\tau_1)^{\kappa} \tag{3.4}$$

where κ is an even integer lying in the interval $(s, s+2]$. It is easily verified algebraically that $E(\hat{\tau}_n)^{\kappa}$ remains bounded, so $P\{\bar{\tau}_n \leq 0\} = o(n^{-s/2})$. Then,

$$\begin{aligned} \varepsilon_n^{\ell}(p) &= -G_n(z(p+1-\alpha)) + o(n^{-s/2}) \\ \varepsilon_n^{r}(p) &= G_n(z(p)) + o(n^{-s/2}) \\ \varepsilon_n(p) &= G_n(z(p+1-\alpha)) - G_n(z(p)) + o(n^{-s/2}) \quad , \end{aligned} \tag{3.5}$$

where $G_n(x) = F_n(x) - \Phi(x)$, provided $E|\tau_n|^{s+2} < \infty$. Analogous expressions hold for the errors for the other confidence intervals. Thus, the discussion of confidence interval error leads to study of asymptotic expansions for $G_n(x)$.

4. Edgeworth Expansions for Smooth Statistics

BHATTACHARAYA and GHOSH (1978) have recently, shown that the "delta method" for deriving Edgeworth expansions is rigorously correct for a wide class of statistics. To be precise, suppose that $\{V_n: n \geq 0\}$ is a sequence of i.i.d. m-dimensional r.v.'s, and let $f_1, \ldots, f_k$ be real-valued Borel measurable functions on R^m. Put

$$U_i = (f_1(V_i), \ldots, f_k(V_i))$$

$$\mu = EU_i$$

and let $H_0, H_1, \ldots, H_s$ be real-valued functions on R^k such that H_i is continuously differentiable of order $s+2-i$ on a neighborhood of μ. The objective is to establish an asymptotic expansion for the distribution of

$$(4.1) \qquad \Lambda_n = n^{1/2}\left(\sum_{k=0}^{s} n^{-k/2}[H_k(\bar{U}_n) - H_k(\mu)]c_n\right) + a_n$$

where

$$a_n = \sum_{k=1}^{s} n^{-k/2} a_{k,s} + o(n^{-s/2}) \quad ,$$

and

$$c_n = 1 + \sum_{k=2}^{s+1} n^{-k/2} c_{k,s} + O(n^{-(s+1)/2})$$

are sequences of deterministic constants. The form of $n^{-1/2} \Lambda_n$ outside a neighborhood of μ can be taken as an arbitrary real-valued measurable function of $\bar{U}_n$ (the constants a_n, c_n add a flexibility which will be necessary later; see (7.2) and (8.3), for example).

We shall henceforth assume, in our study of Λ_n, that the covariance matrix Σ associated with U is non-singular. This can be done, without loss of generality, by replacing $1, f_1, \ldots, f_k$ by a maximal (in terms of number of elements) collection $1, f_{i_1}, \ldots, f_{i_p}$ of functions linearly independent as elements of the L^2 space of r.v.'s (see [1], p. 442, for details).

The "delta method" begins by expanding $H_i(u)$ to $(s+1-i)$ terms, as a Taylor series about $u = \mu$. This yields a polynomial $H_{s,i}(u)$ of degree $(s+1-i)$ and gives rise to a differential approximation $\Lambda_{s,n}$ of Λ_n as follows:

$$(4.2) \qquad \Lambda_{s,n} = n^{1/2}\left(\sum_{k=0}^{s} n^{-k/2}[H_{s,k}(\bar{U}_n) - H_{s,k}(u)]c_n\right) + a_n \; .$$

This can be re-written in the form

$$(4.3) \qquad \Lambda_{s,n}(\hat{U}_n) = \sum_{k=0}^{s} n^{-k/2} P_{s,k}(\hat{U}_n) + o(n^{-s/2})$$

where $P_{s,k}$ is a polynomial of degree $k+1$ and $o(n^{-s/2})$ is independent of $\bar{U}_n$. Assuming that U_1 has sufficiently many finite moments, the j'th cumulant $\kappa_{j,n}$ of $\Lambda_{s,n}$ has the form

$$(4.4) \qquad \kappa_{j,n} = \sum_{k=0}^{s} n^{-k/2} b_{j,k} + o(n^{-s/2})$$

$$= \tilde{\kappa}_{j,n} + o(n^{-s/2}) .$$

Note that $\tilde{\kappa}_{j,n}$ depends exclusively on moments of order less th[illegible] equal to $s+1$. Let σ^2 be the variance of $P_{s,0}(\hat{U}_n)$, which we [illegible] henceforth assume to be positive and finite. A reasonable approximation to the characteristic function of $\Lambda_{s,n}$ (and hence Λ_n) is therefore

$$(4.5) \qquad \exp(-\sigma^2 t^2/s) \exp(1 + \sum_{j=1}^{s+2} \frac{(it)^j}{j!} (\tilde{\kappa}_{j,n} - b_{j,0})) .$$

Expanding the second exponential yields the expression

$$(4.6) \qquad \exp(-\sigma^2 t^2/2) \left[1 + \sum_{r=1}^{s} n^{-r/2} \pi_r(it)\right] + o(n^{-s/2})$$

$$= \hat{\phi}_{s,n}(it) + o(n^{-s/2})$$

where $\pi_1, \ldots, \pi_s$ are polynomials not depending on n. The Fourier transform $\hat{\phi}_{s,n}$ corresponds to the signed measure

$$(4.7)\qquad \phi_{s,n}(v)dv = \left[1 + \sum_{r=1}^{s} n^{-r/2}\, \pi_r(-d/dv)\right] \phi(v/\sigma)\, dv/\sigma$$

which is the formal Edgeworth expansion of the distribution of Λ_n.

(4.8) THEOREM i) Suppose that $E\|U_n\|^3 < \infty$. Then,

$$(4.9)\qquad P\{\Lambda_n \leq x\} = \int_{-\infty}^{x} \phi(v/\sigma)\, dv/\sigma + o(n^{-1/2})$$

where $o(n^{-1/2})$ is uniform in x.

ii) Suppose that $E\|U_n\|^{s+2} < \infty$, and that $U_1 + \cdots + U_\ell$ has a non-zero Lebesgue density component (in R^k) for some ℓ. Then,

$$(4.10)\qquad P\{\Lambda_n \in B\} = \int_B \phi_{s,n}(v)dv + o(n^{-s/2})\quad .$$

where $o(n^{-s/2})$ is uniform over all Borel sets B. The function $\phi_{s,n}$ can be calculated via the "delta method" (4.2) through (4.7).

Although the proof given in [1] restricts its attention to the case where $a_n = 0$, $c_n = 1$, and $H_k \equiv 0$ for $k \geq 1$, the argument readily extends to the more general situation considered here, the only complication being additional notational complexity.

In some related work, CHIBISOV (1972) proved Theorem 4.8 (ii) in the case where T_n was of the polynomial form (4.3) (no identification of $\phi_{s,n}$ as the expansion obtained via the "delta method" was made, however). An extension to the general non-polynomial case was effected via the following "perturbation" theorem (see [4], p. 629).

(4.11) THEOREM. Suppose that $\Lambda_n' = \Lambda_n + n^{-(s+1)/2} \chi_n$ where Λ_n satisfies the assumptions of Theorem 4.8(ii) and $P\{|\chi_n| > n^{1/2} \rho_n\} = o(n^{-s/2})$ for some sequence $\rho_n \to 0$. Then,

$$P\{\Lambda_n' \leq x\} = \int_{-\infty}^{x} \phi_{s,n}(v)dv + o(n^{-s/2})$$

where $o(n^{-s/2})$ is uniform in x.

It is clear that the $\phi_{s,n}$ of Theorem 4.11 must be that obtained via the "delta method".

We remark that the density assumption on the U_i's in Theorem 4.8(ii) follows if V_1 has a Lebesgue density component which is positive on an open set where $1, f_1, \ldots, f_k$ are linearly independent as continuous functions (see Lemma 2.2, [1]). Also, we note that the moment conditions in Theorem 4.8 are norm independent, due to the fact that all norms on finite-dimensional spaces are equivalent.

Because of the potentially large number of derivatives required in calculating the expansions $H_{s,k}(u)$, it is convenient to consider a modification of the "delta method". Towards this end, let $\theta(j;u)$ be polynomials of degree $p(s+1)$, and set

$$\theta_n = \sum_{j=0}^{m} n^{-j/2}\, \theta(j;\, \hat{U}_n) \;.$$

(4.12) **PROPOSITION.** Suppose that $R_n = \Lambda^p_{s,n} - \theta_n$ satisfies $n^{s/2} R_n \to 0$ in probability (i.e., $R_n = o_p(n^{-s/2})$). Then, if $E|U_n|^{(s+1)p} < \infty$, we have $E\,\Lambda^p_{s,n} = E\theta_n + o(n^{-s/2})$.

Proof. The remainder R_n can be written in the form

$$R_n = \sum_{j=0}^{s} n^{-j/2} R(j;\hat{U}_n)$$

where $R(j;u)$ are polynomials in u of degree $p(s+1)$. We now show that $R(j;u)$ vanishes for $j \leq s$. Starting with $j = 0$, observe that

$$R(0;\hat{U}_n) \Longrightarrow R(0;N)$$

($\Longrightarrow$ denotes weak convergence), where N is a multivariate normal r.v. with non-singular covariance matrix Σ. Evidently, since $n^{s/2} R_n = o_p(1)$ and $R_n - R(0;\hat{U}_n) = o_p(1)$, it must be that $R(0;N)$ is degenerate at 0. On the other hand, if $R(0;u)$ depends non-trivially on u_i (say), then the Jacobian of the transformation $u \to (u_1, \ldots, u_{i-1}, R(0;u), u_{i+1}, \ldots, u_k)$ is non-singular, and thus it follows that $R(0;N)$ has a Lebesgue density. This contradiction forces $R(0;u)$ to vanish identically. Repeating the argument s more times proves that

$$R_n = \sum_{j=s+1}^{s} n^{-j/2} R(j;\hat{U}_n)\ .$$

Under the moment conditions given here, $ER(j;\hat{U}_n) \to ER(j;N)$ and consequently $ER_n = o(n^{-s/2})$, proving our result. ∎

Our final goal in this section is to show that the Edgeworth expansion (4.5) remains valid, in a certain sense, when the density assumption on the distribution of U is dropped. Let $C_b^\infty(R)$ be the class of all bounded infinitely differentiable functions and take $C_*^\infty(R) = \{f : D^n f \in C_b^\infty(R), \text{ for all } n\}$ $(D = d/dx)$. The class $C_*^\infty(R)$ includes the trigonometric functions $\sin(tx)$, $\cos(tx)$, as well as the Schwartz class $\underline{S}$ (see BHATTACHARAYA and RAO (1976), p. 257). We first need the following proposition.

(4.13) **PROPOSITION.** (i) Let $f \in C_*^\infty(R)$, and suppose α is a multiindex (i.e., a non-negative integral vector) with $|\alpha| \equiv \Sigma\alpha_i \leq m+2$. Then, if $E\|U_m\|^{m+2} < \infty$, there exists a multivariate Edgeworth expansion $\xi_{m,n}$ of the distribution U such that

$$(4.14) \qquad E(\hat{U}_n^\alpha f(p\cdot\hat{U}_n)) = \int u^\alpha f(p\cdot u)\, \xi_{m,n}(u)du + o(n^{-m/2})$$

holds, for any vector p.

(ii) If $E\|U_m\|^{m+2} < \infty$, then the change-of-variables formula

$$(4.15) \qquad \int f(\Lambda_{m,n}(u))\, \xi_{m,n}(u)du = \int f(y)\, \phi_{m,n}(y)dy + o(n^{-m/2})$$

holds for all bounded measurable f.

Proof. (i) We use Theorem 3.6 of [14], and observe that boundedness of the derivatives of f implies that

$$D^{\beta}(u^{\alpha} f) = O(\|u\|^{m+2})$$

($D^{\beta} = D_1^{\beta_1} \cdots D_k^{\beta_k}$ where $D_j = \partial/\partial x_j$) for all multiindices β with $|\beta| \le m$. This is sufficient for (i), in the presence of $E\|U_n\|^{m+2} < \infty$.

(ii) Lemma 2.1 of [1] proves that (4.15) holds uniformly over all indicator functions f. For an arbitrary bounded f, approximate f by a finite linear combination of indicators $f_{j,n}$ such that

$$\left|f(x) - \Sigma\, d_{j,n}\, f_{j,n}(x)\right| < 2^{-n} \qquad \text{for all } x .$$

Then, letting $\phi(f)$ be the difference between the two integrals in (4.15), and using the Hahn decomposition on the signed measure $\phi(\cdot)$, shows that

$$\begin{aligned} |\phi(f)| &\le 2^{-n}\left(\int |\xi_{m,n}(u)|\,du + \int |\phi_{m,n}(y)|\,dy\right) + \left|\Sigma d_{j,n}\, \phi(f_{j,n})\right| \\ &\le O(2^{-n}) + \sup|f(x)| \cdot o(n^{-s/2}) , \end{aligned}$$

where the uniformity over indicators is used in the final step. ∎

Our next theorem shows that Theorem 4.8 continues to hold, in expectation, when the density assumption on U is deleted. We impose rather strong assumptions on U, and the class of test functions f allowed, in order to simplify the exposition.

(4.16) THEOREM. Suppose that U_n has finite moments of all orders. Then, for any s,

$$\text{(4.17)} \qquad Ef(\Lambda_n) = \int f(y)\,\phi_{s,n}(y)dy + o(n^{-s/2})$$

for all $f \in C_s^\infty(R)$. The function $\phi_{s,n}$ can be identified through the "delta method."

<u>Proof</u>. First, observe that for any $\varepsilon > 0$, there exists $K > 0$ such that

$$\text{(4.18)} \qquad P\{|\bar{U}_n - \mu| > \varepsilon\} \leq P\{|\hat{U}_n| > K \ln n\} .$$

The probability on the right-hand side of (4.18) is $o(n^{-s/2})$ (see Corollary 17.12 of [2]); and hence

$$Ef(T_n) = E\{f(T_n); |\bar{U}_n - \mu| < \varepsilon\} + o(n^{-s/2}) .$$

Choose ε sufficiently small that $D^{(s+2-k)} H_k(u)$ is continuous on an ε-neighborhood of μ for all k. Expanding Λ_n on $\{|\bar{U}_n - \mu| < \varepsilon\}$ yields

$$(4.19) \qquad \Lambda_n = \Lambda_{s,n} + n^{-(s+1)/2} \sum_{k=0}^{s} c_n(\hat{U}_n \cdot \nabla)^{s+2-k} H_k(\eta_{k,n})/(s+2-k)!$$

where $\nabla = (D_1, \ldots, D_k)$ and $\|\eta_{k,n}-\mu\| < \varepsilon$. Note that $H_k(\eta_{k,n})$ are bounded r.v.'s in (4.19).

Thus, we can write $f(T_n)$ on $\{\|\bar{U}_n-\mu\| < \varepsilon\}$ as

$$f(\Lambda_n) = \sum_{k=0}^{s} (\Lambda_n - P_{o,n}(\hat{U}_n))^k \, (D^k f) \, (P_{o,n}(\hat{U}_n))/k!$$

$$+ (\Lambda_n - P_{o,n}(\hat{U}_n))^{s+1} \, (D^{s+1} f)(\eta_n)/(s+1)!$$

which evidently can be re-written as

$$f(\Lambda_n) = \sum_{j=0}^{\ell} n^{-j/2} \, \Gamma_{j,n}(\hat{U}_n;\, \eta_{k,n};\, \eta_n) \; .$$

For $j \leq s$, $\Gamma_{j,n}$ has the form $\hat{U}_n^{\alpha} \, g(p \cdot \hat{U}_n)$ $(g \in C_*^{\infty}(R))$ whereas for $j > s$, $\Gamma_{j,n}$ is the product of functions of this form with bounded functions. Thus, for $j \leq s$,

$$E\{\Gamma_{j,n};\ \|\bar{U}_n-\mu\| < \varepsilon\}$$

$$= E\Gamma_{j,n} + o(n^{-s/2}) = \int \Gamma_{j,n}(u) \, \xi_{s,n}(u) du + o(n^{-s/2}) \; ,$$

the first equality by uniform integrability of $\{\Gamma_{j,n}\}$, the second by Proposition 4.13(i). A similar argument for $j > s$ shows that

$$E\{\sum_{j=s+1}^{\ell} n^{-j/2}\,\Gamma_{j,n};\ |\bar{U}_n-\mu| < \epsilon\} = o(n^{-s/2})$$

and hence

$$Ef(\Lambda_n) = \int \sum_{j=0}^{s} n^{-j/2}\,\Gamma_{j,n}(u)\ \xi_{s,n}(u)du + o(n^{-s/2})$$

$$= \int \Lambda_{s,n}(u)\ \xi_{s,n}(u)du + o(n^{-s/2}) \quad .$$

Applying Proposition 4.13(ii) completes the proof of (4.17). The identification of $\phi_{s,n}(y)$ as that derived from the "delta method" follows from a proof identical to that found on pages 445-6 of [1]. ∎

We remark that an immediate consequence of Theorem 4.16 is that, under the assumptions stated, the characteristic function of Λ_n can be expanded as

$$E\,\exp(it\,\Lambda_n) = \hat{\phi}_{s,n}(it) + o(n^{-s/2}) \quad .$$

5. Edgeworth Expansions for Ratio Estimator Pivots

As Section 4 illustrates, the key to obtaining Edgeworth expansions is the calculation of cumulants (see (4.4)) of the differential approximation $\Lambda_{s,n}$. The required moments will be derived from Proposition 4.12. In this section, we will calculate Edgeworth expansions for t_n and t_n^* to order n^{-1}. This represents a

different approach from that of GEARY (1947) and GAYEN (1949), who formally expanded the distribution of the pivot t_n (for the classical case where $\tau_n \equiv 1$) in Charlier-type series.

We start by observing that the pivotal quantities t_n and t_n^* are invariant to the transformation $(Y_i,\tau_i) \to (\alpha Y_i, \alpha\tau_i) = (Y_i',\tau_i')$ for $\alpha \neq 0$. In particular, by taking $\alpha = 1/\sigma(Z)$, we can assume throughout our calculations, via a passage to (Y_i',τ_i'), that $\sigma(Z) = 1$. However, in stating our final conclusions, dependence on $\sigma(Z)$ will be made explicit.

Our first order of business is to expand t_n and t_n^* in differential-type approximations $t_{2,n}$ and $t_{2,n}^*$. Set $V_k = (Y_k,\tau_k)$ and let $f_1(v) = v_1$, $f_2(v) = v_2$, $f_3(v) = v_1^2$, $f_4(v) = v_1v_2$, $f_5(v) = v_2^2$. Observe that $r_n - r = \bar{Z}_n/\bar{\tau}_n$ and that $I_{\{v_n>0,\bar{\tau}_n\neq 0\}}$ is identically 1 on a neighborhood of μ. Thus,

$$(5.1)\qquad t_n^* = \hat{Z}_n\left(1 - \frac{1}{2}(v_n^*-1) + \frac{3}{4}(v_n^*-1)^2 + o_p(n^{-1})\right)$$

where

$$(5.2)\qquad v_n^*-1 = \bar{W}_n - 2\frac{\bar{Z}_n}{\bar{\tau}_n}\sum_{i=1}^{n}\frac{Y_i\tau_i}{n} + \frac{\bar{Z}_n}{\bar{\tau}_n}(r_n+r)\sum_{i=1}^{n}\frac{\tau_i^2}{n}$$

and $W_i = Z_i^2-1$. Expanding $1/\bar{\tau}_n$ in a Taylor series about $1/E\tau$, we obtain

$$(5.3) \qquad \bar{Z}_n/\bar{\tau}_n = \bar{Z}_n/E\tau - (\bar{Z}_n\hat{\tau}_n)/n^{1/2}\,(E\tau)^2 + O_p(n^{-2})$$

where we use $O_p(n^{-k/2})$ to represent a sequence of r.v.'s K_n such that $n^{k/2}\,K_n$ remains bounded in probability (see p. 8 of SERFLING (1980)). Also,

$$(5.4) \qquad 2\sum_{i=1}^{n} Y_i\tau_i/n - (r_n+r)\sum_{i=1}^{n}\tau_i^2/n$$

$$= 2\sum_{i=1}^{n} Z_i\tau_i/n - (\bar{Z}_n/\bar{\tau}_n)\sum_{i=1}^{n}\tau_i^2/n\ .$$

Relations (5.1) through (5.4) imply that

$$(5.5) \qquad t_n^* = Z_n\{1 - \hat{W}_n/2n^{1/2} + \alpha\hat{Z}_n/n^{1/2}$$

$$- \hat{Z}_n\hat{Q}_n/n + 3(\hat{W}_n - 2\alpha\hat{Z}_n)^2/4n\} + o_p(n^{-1})$$

where

$$Q_i = 2\alpha\ \tau_i/E\tau - 2Z_i\tau_i/(nE\tau) + Z_i\delta$$

and $\alpha = EZ\tau/(\sigma(Z)E\tau)$ and $\delta = E\tau^2/(E\tau)^2$. In order to evaluate the cumulants required, some moment identities are needed.

(5.6) THEOREM (i) Let $\{A_n;\ n \geq 0\}$ be a sequence of i.i.d. r.v's with $EA_n^k < \infty$ for all k. Then,

(a) $E\hat{A}_n^3 = n^{-1/2}\, E\hat{A}_1^3$

(b) $E\hat{A}_n^4 = 3(E\hat{A}_1^2)^2 + n^{-1/2}\, E\hat{A}_1^4 - 2n^{-1/2}(E\hat{A}_1^2)^2$

(c) $E\hat{A}_n^5 = 10\, n^{-1/2}(E\hat{A}_1^3)(E\hat{A}_1^2) + o(n^{-1})$

(d) $E\hat{A}_n^6 = 15(E\hat{A}_1^2)^3 + o(n^{-1/2})$

(e) $E\hat{A}_n^{2k+1} = o(n^{-1/2})$.

(ii) Suppose that $\{(A_n, B_n);\ n \geq 1\}$ is a sequence of i.i.d. r.v.'s with $EA_n^k < \infty$, $EB_n^k < \infty$ for all k. Then,

(f) $E\hat{A}_n^2\hat{B}_n = n^{-1/2}\, E\hat{A}_1^2\hat{B}_1$

(g) $E\hat{A}_n^3\hat{B}_n = 3(E\hat{A}_1\hat{B}_1)(E\hat{A}_1^2) + o(n^{-1/2})$

(h) $E\hat{A}_n^4\hat{B}_n = 4\, n^{-1/2}(E\hat{A}_1^3)\, E\hat{A}_1\hat{B}_1 + 6n^{-1/2}(E\hat{A}_1^2\hat{B}_1)\, E\hat{A}_1^2 + o(n^{-1})$

(i) $E\hat{A}_n^4\hat{B}_n^2 = 3(E\hat{A}_1^2)(E\hat{B}_1^2)^2 + 12(E\hat{A}_1\hat{B}_1)^2\ E\hat{B}_1^2 + O(n^{-1/2})$

(j) $E\hat{A}_n^5\hat{B}_n = 15(E\hat{A}_1\hat{B}_1)\ (E\hat{A}_1^2)^2 + O(n^{-1/2})$

(k) $E\hat{A}_n^k\hat{B}_n^j = O(n^{-1/2})$ if $k+j$ is odd .

Proof. For (i), observe that exchangeability of the sequence $\{A_n\colon n \geq 0\}$ provides a recursion in k, namely

$$E\Big(\sum_{i=1}^{n} A_i'\Big)^{k+1} = E\Big(\sum_{i=1}^{n} A_i'\Big)\Big(\sum_{i=1}^{n} A_i'\Big)^k$$

$$= nE\Big(A_n'\Big(\sum_{i=1}^{n} A_i'\Big)^k\Big)$$

$$= n \sum_{\ell=1}^{k} \binom{k}{\ell} E(A_i')^{\ell+1}\ E\Big(\sum_{j=1}^{n-1} A_j'\Big)^{k-\ell}$$

where $A_i' = A_i - EA_1$. Solving the recursion with initial condition $EA_n' = 0$ proves (i). For (ii), apply (i) to $A_n(s,t) = sA_n + tB_n$. Both sides of equations (a) through (e) are then polynomials in (s,t). Identifying coefficients yields results (f) through (k). ∎

Now, let $\beta = EZ_1^3/\sigma^3(Z_1)$, $\lambda = (EZ_1^4/\sigma^4(Z_1)) - 3$, and $\gamma = EZ_1^2\tau_1/(\sigma^2(Z_1)E\tau_1)$. Then,

$$
\begin{aligned}
&E\hat{Z}_1\,\hat{W}_1 = \beta\ , && E\hat{Z}_1^2\,\hat{W}_1 = \lambda+2 \\
(5.7)\qquad & && \\
&E\hat{Z}_1\,\hat{Q}_1 = \delta + 2\alpha^2 - 2\gamma\ , && E\hat{W}_1^2 = \lambda+2
\end{aligned}
$$

In view of Proposition 4.12, (5.5) through (5.7) provide the asymptotic expansion

$$Et^*_{2,n} = (\alpha-\beta/2)n^{-1/2} + o(n^{-1})\ .$$

Similar reasoning on the higher moments proves that

$$E(t^*_{2,n})^2 = 1 + (6\alpha^2 + 6\gamma + 2\beta^2 - 3\delta - 10\alpha\beta)n^{-1} + o(n^{-1})$$

$$E(t^*_{3,n})^3 = (9\alpha - 7\beta/2)n^{-1/2} + o(n^{-1})$$

$$E(t^*_{2,n})^4 = 3 + (120\,\alpha^2 + 60\,\gamma + 28\,\beta^2 - 30\,\delta - 140\,\alpha\beta - 2\lambda - 6)n^{-1} + o(n^{-1})\ .$$

The relevant expansions for t_n may be easily obtained by using the relation $t_n = t^*_n(1 - 1/2n + O(n^{-2}))$. Consequently,

$$Et_{2,n} = Et^*_{2,n} ,$$

$$Et^2_{2,n} = E(t^*_{2,n})^2 - n^{-1} ,$$

$$Et^3_{2,n} = E(t^*_{2,n})^3 ,$$

and

$$Et^4_{2,n} = E(t^*_{2,n})^4 - 6n^{-1} ,$$

up to terms of order $o(n^{-1})$.

Note that in the classical case where $\tau_n \equiv 1$, we have $\alpha = 0$ and $\gamma = \delta = 1$. The moment formulas for t_n, when appropriately simplified, are then in agreement with those found in [11] and [12]. It should also be noted that in the classical case, the approximate skewness $E(t_{2,n})^3$ $((E(t^*_{2,n})^3)$ of $t_n(t^*_n)$ is $-7\beta/2$. This verifies the empirical observation that positive skewness in the distribution of Z_i leads to negative skewness in the pivots t_n and t^*_n (see SOPHISTER (1928), and NEYMAN and PEARSON (1928)).

The appropriate cumulants $\tilde{\kappa}_{j,n}$ of t_n are given by

$$\tilde{\kappa}_{j,n} = \sum_{k=0}^{2} n^{-k/2} b_{j,k} + o(n^{-1}) \tag{5.8}$$

where

$$b_{1,1} = \alpha - \beta/2$$

$$b_{2,0} = 1$$

$$b_{2,2} = 6\gamma + 5\alpha^2 + 7\beta^2/4 - 3\delta - 9\alpha\beta - 1$$

$$b_{3,1} = 6\alpha - 2\beta$$

$$b_{4,2} = 24\gamma + 60\alpha^2 + 12\beta^2 - 12\delta - 60\alpha\beta - 2\lambda - 6$$

and all other $b_{j,k}$ are zero. For t_n^*, the $b_{j,k}^*$ of the corresponding cumulants $\tilde{\kappa}_{j,n}^*$ satisfy $b_{j,k}^* = b_{j,k}$, excepting that $b_{2,2}^* = b_{2,2}+1$.

The following distribution function $\Psi_{2,n}(x)$ is obtained from $\tilde{\kappa}_{j,n}$ in the same way as passage was made from (4.4) to (4.7):

$$
\begin{aligned}
(5.9) \quad \Psi_{2,n}(x) = \Phi(x) &- b_{1,1}\, \phi(x)/n^{1/2} \\
&- (b_{2,2} + b_{1,1}^2)x\, \phi(x)/2n \\
&+ b_{3,1}(1-x^2)\, \phi(x)/6n^{1/2} \\
&- (b_{4,2} + 4b_{1,1}\, b_{3,1})\, (x^3-3x)\, \phi(x)/24n \\
&- b_{3,1}^2(x^5 - 10x^3 + 15x)\, \phi(x)/72n
\end{aligned}
$$

Also, let $\Psi_{2,n}^*(x)$ be the function obtained from (5.9) by substituting $b_{j,k}^*$ in place of $b_{j,k}$.

(5.10) **THEOREM** (i) If $E(Y_n^6 + \tau_n^6) < \infty$, then

$$P\{t_n \leq x\} = \Phi(x) + O(n^{-1/2})$$

where $O(n^{-1/2})$ is uniform in x.

(ii) Suppose that $E(|Y_n|^k + |\tau_n|^k) < \infty$ for all k. Then,

$$Ef(t_n) = \int f(y)\, \Psi_{2,n}(dy) + o(n^{-1})$$

for all $f \in C_*^\infty(R)$.

(iii) Suppose $E(Y_n^8 + \tau_n^8) < \infty$, and that (Y_1, τ_1) has a distribution with a Lebesgue density component which is positive on some open set in the plane. Then,

$$P\{t_n \in B\} = \int_B \Psi_{2,n}(dy) + o(n^{-1})$$

where $o(n^{-1})$ is uniform over all Borel sets B.

(iv) Suppose $\tau_n \equiv 1$. Then, if $EY_n^8 < \infty$, and if the distribution of Y_n has a Lebesgue density component which is positive on some interval, the analogue of (iii) above holds. The function $\Psi_{2,n}$ is obtained from (5.9) by formal substitution.

(v) Results (i) to (iv) are valid for t_n^* under the same assumptions as for t_n, provided that $\Psi_{2,n}^*$ is substituted in place of $\Psi_{2,n}$.

Proof. The functions $f_1, \ldots, f_5$, being distinct polynomials, are linearly independent so Theorems 4.8 and 4.16 can be applied,

yielding (i) to (iii). Part (iv) is handled as a special case, by setting $V_i = Y_i$, and $f_1(v) = v$, $f_2(v) = v^2$, and applying the same argument as for (iii). ∎

As previously mentioned, a particularly important application of ratio estimation lies in the domain of ergodic analysis of regenerative stochastic processes. It frequently occurs that the regenerative sequence $\{(Y_i, \tau_i);\ i \geq 1\}$ constructed is such that Y_1 has a Lebesgue density component, whereas τ_1 is a lattice r.v. For example, this is the case that arises when $\{X_t;\ t \geq 0\}$ is a continuous time process constructed from a discrete time regenerative process $\{X'_n\}$ via the formula

$$X_t = \sum_{n=1}^{\infty} X'_n \, I_{\{n \leq s < n+1\}}(t) \quad .$$

Our next result addresses this class of processes.

(5.11) THEOREM. Suppose that $E(Y_n^6 + \tau_n^6) < \infty$ and that Z_1 has a distribution with a Lebesgue density component which is positive on an interval. Then,

$$P\{t_n \leq x\} = \Psi_{1,n}(x) + o(n^{-1/2})$$

$$P\{t_n^* \leq x\} = \Psi_{1,n}(x) + o(n^{-1/2})$$

uniformly in x, where $\Psi_{1,n}(x)$ is obtained from $\Psi_{2,n}(x)$ by deleting terms with coefficient n^{-1}.

Proof. The pivot t_n can be expanded as

$$(5.12)\qquad t_n = I_{\{v_n>0,\bar{\tau}_n\neq 0\}}\left(\hat{Z}_n - \frac{\hat{W}_n\hat{Z}_n}{2n^{1/2}} + \frac{\alpha\hat{Z}_n^2}{n^{1/2}}\right)\left(1 + O\left(\frac{1}{n}\right)\right) + \chi_n/6n$$

where $O(1/n)$ is deterministic and

$$\chi_n = I_{\{v_n>0,\bar{\tau}_n\neq 0\}}(\hat{U}_n\cdot\nabla)^3\, H(\xi_n)$$

and H corresponds to t_n via (4.1). Now, observe that Theorem 4.8(ii), with $s = 1$, is applicable to the first term in (5.12).

Select ε small enough so that $(D^\tau H)(u)$ is bounded for $\|u-\mu\| \leq \varepsilon$ for all multiindices τ with $|\tau| = 3$. Let $\rho_n = K(\ln n/n)^{1/2}$ for K to be chosen later. Then,

$$P\{\|\chi_n\| > \rho_n\, n^{1/2}\}$$

$$\leq P\{\|\chi_n\| > \rho_n\, n^{1/2};\ \|\bar{U}_n-\mu\| \leq \varepsilon\} + P\{\|\bar{U}_n-\mu\| > \varepsilon\}$$

$$\leq P\{\|\chi_n\| > \alpha K \ln n\} + P\{\|\hat{U}_n\| > K \ln n\}\ .$$

Choose K sufficiently small so that (4.18) applies. Thus,

$$P\{\|\chi_n\| > \rho_n n^{1/2}\} = o(n^{-1/2}) ,$$

and hence Theorem 4.11 implies the result for t_n. Precisely the same argument works for t_n^*. ∎

6. <u>Applications to Ratio Estimator Confidence Interval Estimation</u>

In this section, we apply the Edgeworth expansions of Section 5 to analysis of nonparametric ratio estimator confidence intervals.

(6.1) THEOREM (i) Suppose $E(Y_n^6 + \tau_n^6) < \infty$. Then, $\varepsilon_n^{\ell}(p)$, $\varepsilon_n^{r}(p)$, and $\varepsilon_n(p)$ are all $O(n^{-1/2})$, uniformly in p.

(ii) Under the assumptions of Theorem 5.11,

$$\varepsilon_n^{\ell}(p) = p + 1 - \alpha - \Psi_{1,n}(z(p+1-\alpha)) + o(n^{-1/2})$$

$$\varepsilon_n^{r}(p) = \Psi_{1,n}(z(p)) - p + o(n^{-1/2})$$

$$\varepsilon_n(p) = \Psi_{1,n}(z(p+1-\alpha)) - \Psi_{1,n}(z(p)) - (1-\alpha) + o(n^{-1/2})$$

uniformly in p.

(iii) Under either assumptions (iii) or (iv) of Theorem 5.10,

$$\varepsilon_n^{\ell}(p) = p + 1 - \alpha - \Psi_{2,n}(z(p+1-\alpha)) + o(n^{-1})$$

$$\varepsilon_n^r(p) = \Psi_{2,n}(z(p)) - p + o(n^{-1})$$

$$\varepsilon_n(p) = \Psi_{2,n}(z(p+1-\alpha)) - \Psi_{2,n}(z(p)) - (1-\alpha) + o(n^{-1})$$

uniformly in p.

(iv) Results (i) to (iii) are valid for $\varepsilon_n^\ell(p)^*$, $\varepsilon_n^r(p)^*$, and $\varepsilon_n(p)^*$ under the same assumptions as for the t_n errors, provided $\Psi_{2,n}^*$ is substituted in place of $\Psi_{2,n}$.

Proof. The results follow immediately from (3.5), and Theorems 5.10 and 5.11. ∎

These expressions show that under reasonable assumptions the coverage errors $\varepsilon_n(p)$, $\varepsilon_n(p)^*$ are $O(n^{-1/2})$ for $p \neq \alpha/2$, whereas for $p = \alpha/2$, the coverage errors are $O(n^{-1})$. Thus, using confidence intervals based on $p = \alpha/2$ leads to intervals that are asymptotically optimal in the sense of having shortest possible length and most accurate coverage rate. However, it is important to realize that the one-sided coverage errors are $O(n^{-1/2})$ for all p, including $p = \alpha/2$. Hence it must be that $[L_n(\alpha/2), R_n(\alpha/2)]$ (similarly for $[L_n^*(\alpha/2), R_n^*(\alpha/2)]$) achieves coverage error of $O(n^{-1})$ via cancellation of one-sided errors of order $O(n^{-1/2})$. This suggests that a "corrected interval", in the sense of one-sided error, can be obtained by shifting the interval slightly. This is in agreement with parametric confidence interval theory, where intervals tend to be

asymmetric about the point estimate. We shall examine this question further in Section 7.

The coverage errors $\varepsilon_n(\alpha/2)$ and $\varepsilon_n(\alpha/2)^*$ are given by

$$\begin{aligned}\varepsilon_n(\alpha/2) = &-(b_{2,2} + b_{1,1}^2)x_\alpha \, \phi(x_\alpha)/n \\ &-(b_{4,2} + 4b_{1,1}\, b_{3,1})\,(x_\alpha^3 - 3x_\alpha)\,\phi(x_\alpha)/12n \\ &-b_{3,1}^2(x_\alpha^5 - 10x_\alpha^3 + 15x_\alpha)\,\phi(x_\alpha)/36n\end{aligned}$$

$$\varepsilon_n(\alpha/2)^* = \varepsilon_n(\alpha/2) - x_\alpha \, \phi(x_\alpha)/n$$

where $x_\alpha = z(1 - \alpha/2)$. Recalling the definition of the $b_{j,k}$'s (see (5.7)) we see that $\varepsilon_n(\alpha/2)$ and $\varepsilon_n(\alpha/2)^*$ have a tendency to be negative, particularly if the Z_i's are highly skewed (i.e., β^2 is large). This tendency for nonparametric confidence intervals to undercover has been exhibited empirically; see IGLEHART (1975), for example. The procedure of Section 8 will attempt to deal with this coverage rate problem.

Note that the t_n coverage error is always biased upwards from that of t_n^* by an amount $z(1-\alpha/2)\,\phi(z(1-\alpha/2))/n$. This is an attractive property of t_n, in comparison to t_n^*, in view of the undercoverage mentioned above. The cost associated with using t_n, rather than t_n^*, is that the t_n interval is longer, asymptotically, by an amount $\sigma(Z)\, z(1-\alpha/2)/((E\tau)\, n^{3/2})$.

A similar analysis can be performed for the intervals $[L_n'(\alpha/2), R_n'(\alpha/2)]$. PEISER (1943) showed that

$$(6.2) \qquad z_{n-1}(p) = z(p) + (z^3(p) + z(p))/4n + o(n^{-1}) \quad .$$

Thus, using the uniformity in x of the expansion $\Psi_{2,n}(x)$, we get

$$(6.3) \quad \varepsilon_n(\alpha/n)' = \Psi_{2,n}(z_{n-1}(1-\alpha/2)) - \Psi_{2,n}(z_{n-1}(\alpha/2)) - (1-\alpha) + o(n^{-1})$$

$$= \varepsilon_n(\alpha/2) + (x_\alpha^3+x_\alpha)\ \phi(x_\alpha)/2n + o(n^{-1})$$

where $x_\alpha = z(1-\alpha/2)$. Thus, the coverage rate for the interval $[L_n'(\alpha/2),\ R_n'(\alpha/2)]$ tends to be larger than that of t_n, by an amount $(x_\alpha^3+x_\alpha)\ \phi(x_\alpha)/2n$. For highly skewed populations, this gives intervals based on Student t-quantiles an advantage over those based on normal quantiles. The use of Student t-quantiles comes at the cost of an interval which is longer by an amount $\sigma(Z)(x_\alpha^3+x_\alpha)/(E\tau)\ n^{3/2}$, however.

Note that for samples from populations with normal Y_i and $\tau_i \equiv 1$, $\varepsilon_n(\alpha/2)' = o(n^{-1})$, as expected.

7. Johnson's Pivotal Transformation

In Section 6, it was shown that under reasonably general assumptions, the one-sided coverage errors for t_n and t_n^* are of order $O(n^{-1/2})$. These errors arise due to asymmetry effects related to skewness and ratio estimator bias. In a recent paper JOHNSON (1978) considered, in the case where $\tau_n \equiv 1$, a transformation of the pivot t_n derived on the basis of Cornish-Fisher expansions (see CORNISH and FISHER (1937) for a discussion of these expansions). Empirical evidence collected by Johnson indicated that the

transformation led to intervals that reflected the "correct" degree of asymmetry. We now investigate the pivotal transformation of Johnson, using the machinery developed in Section 4.

Consider the sequence

$$(7.1) \qquad T_n = t_n + \theta_n n^{-1/2} + \rho_n(t_n)^2 n^{-1/2}$$

where $\theta_n = \theta(\bar{U}_n, v_n)$, $\rho_n = \rho(\bar{U}_n, v_n)$ and $\theta(\cdot)$, $\rho(\cdot)$ are functions analytic on a neighborhood of $(\mu, \sigma(z))$. Let $\theta = \theta(\mu, \sigma^2(Z))$, $\rho = \rho(\mu, \sigma^2(Z))$, and observe that

$$(7.2) \qquad T_n = \hat{Z}_n + \left(\alpha\hat{Z}_n - \hat{W}_n\hat{Z}_n/2 + \theta + \rho Z_n^2\right) n^{-1/2} + n^{-1} \chi_n \quad .$$

$$= T_n' + n^{-1} \chi_n$$

where χ_n is $O_p(1)$.

We now use Proposition 4.12, Theorem 5.6, and relation (5.7) to obtain the cumulant expressions

$$\kappa_1(T_{1,n}) = (-\beta/2 + \alpha + \theta + \rho)\, n^{-1/2} + O(n^{-1})$$

$$\kappa_2(T_{1,n}) = 1 + O(n^{-1})$$

$$\kappa_3(T_{1,n}) = (-2\beta + 6\alpha + 6\rho)\, n^{-1/2} + O(n^{-1}) \quad .$$

Observe that by setting $\theta = \beta/6$, $\rho = \beta/3 - \alpha$, all three cumulants above are reduced to $O(n^{-1})$. This suggests setting $\theta_n = \beta_n/6$, $\rho_n = \beta_n/3 - \alpha_n$ where

(7.3) $$\beta_n = I_{\{v_n>0,\bar{\tau}_n\neq 0\}} \sum_{i=1}^{n} (Y_i - r_n\tau_i)^2/(nv_n^{3/2})$$

$$\alpha_n = I_{\{v_n>0,\bar{\tau}_n\neq 0\}} \sum_{i=1}^{n} (Y_i - r_n\tau_i)\ \tau_i/(nv_n^{1/2}\ \bar{\tau}_n)\ .$$

Let T_n^* be defined through (7.1) and (7.3), substituting t_n^* and v_n^* for t_n and v_n.

(7.4) THEOREM (i) Suppose that $E(|Y_n|^k + |\tau_n|^k) < \infty$ for all k. Then,

$$Ef(T_n) = \int f(y)\ \phi(y)dy + o(n^{-1/2})$$

for all $f \in C_b^\infty(R)$.

(ii) If $E(|Y_n|^9 + |\tau_n|^9) < \infty$, and if the density assumption of Theorem 5.11 holds, then

$$P\{T_n \leq x\} = \Phi(x) + o(n^{-1/2})$$

uniformly in x.

(iii) Results (i) and (ii), under the assumptions stated, are valid for T_n^*.

Proof. First, observe that χ_n (see (7.2)) is the remainder term from Taylor's theorem for T_n. On the set $\{v_n > 0, \bar{\tau}_n \neq 0\}$, χ_n has the form

$$\chi_n = (\hat{U}_n \cdot \nabla)^3 H_n(\xi_n)/6$$

when $H_n(\xi_n)$ is bounded on $\{|\bar{U}_n - \mu| < \varepsilon\}$. Now, apply Theorems 4.8(ii) and 4.16, as in the proof of Theorem 5.11, to obtain (ii). For (i), write

$$Ef(T_n) = Ef(T_n') + E\chi_n \, Df(\eta_n)/n + o(n^{-1/2})$$

and argue as in the proof of Theorem 4.16. The proofs for T_n^* can be handled similarly. ∎

We remark that the moment assumptions in Theorem 7.4 follow from the fact that U_i must be expanded to include $Y_i^k \tau_i^j$ with $k+j = 3$, due to the presence of β_n in T_n.

For the classical case where $\tau_n \equiv 1$, the transformed pivots T_n and T_n^* are precisely the statistics suggested by Johnson, up to a term which is $O_p(n^{-1})$. Note that Theorem 7.4 gives rigorous substance to the statement that $T_n(T_n^*)$ "normalizes" $t_n(t_n^*)$ in the sense of creating a r.v. which is closer to a normal. This is not surprising, in light of the fact that Johnson's calculations were based on Cornish-Fisher expansions, which are "normalization" series (see [30], p. 643).

Theorem 7.4 can be easily applied to coverage error asymptotics to yield the following result: If (Y_n, τ_n) satisfies the assumptions of Theorem 7.4(ii), then all the coverage errors (one-sided as well as two-sided) for intervals based on T_n or T_n^*

are $o(n^{-1/2})$ uniformly in the parameter p. Thus, the Johnson pivotal transformation corrects for asymmetry effects.

8. A Second-Order Pivotal Transformation

As discussed in Section 6, nonparametric confidence intervals have a tendency to undercover at small sample sizes. However, the analysis of the symmetric intervals $[L_n(\alpha/2), R_n(\alpha/2)]$ showed that the coverage error is basically determined by the term in n^{-1} of the asymptotic expansion of $P\{t_n \leq x\}$. This suggests that any attempt to correct the coverage rate of the symmetric intervals $[L_n(\alpha/2), R_n(\alpha/2)]$ must deal with higher order error terms than those considered by the Johnson pivotal transformation.

Consider the statistic

$$\hat{T}_n = T_n + \nu_n t_n n^{-1} + \omega_n t_n^3 n^{-1} \tag{8.1}$$

where $\nu_n = \nu(\bar{U}_n) + O_p(n^{-3/2})$, $\omega_n = \omega(\bar{U}_n) + O_p(n^{-3/2})$, and $\nu(\cdot)$, $\omega(\cdot)$ are functions analytic on a neighborhood of μ. Before proceeding with an expansion of $\hat{T}_n$, we state the following approximations:

$$\beta_n = \bar{M}_n - 3\bar{Z}_n\gamma/\sigma(Z) + O_p(n^{-3/2}) \tag{8.2}$$

$$\alpha_n = \bar{N}_n - \bar{Z}_n\delta/\sigma(Z) + O_p(n^{-3/2})$$

where $M_i = Z_i^3/\sigma^3(Z_i)$, $N_i = Z_i\tau_i/(\sigma(Z)\ E\tau)$. Substituting (8.2) into (8.1) yields the following expansion (for the purposes of calculation, we take $\sigma(Z) = 1$):

$$
\begin{aligned}
(8.3)\quad \hat{T}_n = \{\hat{Z}_n(1 &- \tfrac{1}{2}(\hat{W}_n - 2\alpha\hat{Z}_n + \hat{Z}_n\hat{Q}_n\, n^{-1/2})\, n^{-1/2} \\
&+ \tfrac{3}{4}(\hat{W}_n - 2\alpha\hat{Z}_n)^2\, n^{-1}\}\ (1-n^{-1}) \\
&+ \lambda n^{-1/2} + \rho\hat{Z}_n^2(1 - (\hat{W}_n - 2\alpha\hat{Z}_n)n^{-1/2})n^{-1/2} + \hat{M}_n/6n - \hat{Z}_n\gamma/2n \\
&+ (\hat{M}_n/3 - \hat{Z}_n\gamma - \hat{N}_n + \hat{Z}_n\delta)\ \hat{Z}_n^2\, n^{-1} \\
&+ \nu\hat{Z}_n\, n^{-1} + \omega\hat{Z}_n^3\, n^{-1} + O_p(n^{-3/2}) \\
= \hat{T}_n' &+ n^{-3/2}\, \chi_n\ .
\end{aligned}
$$

Proposition 5.12 can be used to calculate the cumulants of $\hat{T}_n'$:

$$
\begin{aligned}
(8.4)\quad \kappa_1(\hat{T}_n') &= O(n^{-3/2}) \\
\kappa_2(\hat{T}_n') &= 1 + (3\delta-7\gamma-9\alpha^2+\alpha\beta-\beta^2/36+7\lambda/3+6+2\nu+6\omega)n^{-1} + O(n^{-3/2}) \\
\kappa_3(\hat{T}_n') &= O(n^{-3/2}) \\
\kappa_4(\hat{T}_n') &= (6\lambda+4\beta^2/3+18+4\alpha\beta+12\delta-24\gamma-3\alpha^2+24\omega)n^{-1} + O(n^{-3/2})\ .
\end{aligned}
$$

Once again, a judicious choice of ν and ω can reduce the order of the above cumulants to $O(n^{-3/2})$. In fact, choosing

$$\nu_n = \gamma_n/2 + 13\beta_n^2/72 - 3/4 - 5\lambda_n/12$$

$$\omega_n = \gamma_n + 3\alpha_n^2/12 - \delta_n/2 + \alpha_n\beta_n/2 - \beta_n^2/18 - 3/4 - \lambda_n/4$$

where

$$\gamma_n = I_{\{v_n>0,\bar{\tau}_n\neq 0\}} \sum_{i=1}^{n} (Y_i - r_n\tau_i)^2 \; \tau_i/(n\bar{\tau}_n \; v_n)$$

$$\lambda_n = I_{\{v_n>0,\bar{\tau}_n\neq 0\}} \sum_{i=1}^{n} (Y_i - r_n\tau_i)^4/(nv_n^2) - 3$$

$$\delta_n = I_{\{\bar{\tau}_n\neq 0\}} \sum_{i=1}^{n} \tau_i^2/\bar{\tau}_n^2$$

yields a pair $\nu = \nu(\mu)$, $\omega = \omega(\mu)$ which reduces the above cumulant expressions (8.4) to $O(n^{-3/2})$.

(8.5) THEOREM (i) Suppose that $E(|Y_n|^k + |\tau_n|^k) < \infty$ for all k. Then,

$$Ef(\hat{T}_n) = \int f(y)\,\phi(y)dy + o(n^{-1}), \qquad \text{for all } f \in C_*^\infty(\mathbb{R}).$$

(ii) If $E(Y_n^{16} + \tau_n^{16}) < \infty$, and if (Y_n, τ_n) satisfies the density assumptions (5.10) (iii) or (iv), then

$$P\{\hat{T}_n \leq x\} = \Phi(x) + o(n^{-1})$$

uniformly in x.

Proof. We apply Theorems 4.8(iii) and 4.16 to $\hat{T}_n'$ (note that now fourth moments of (Y_n, τ_n) must be included in U_n). The perturbation χ_n of (8.3) is a Taylor series remainder similar to that found in (7.2). One then argues as in the proof of Theorem 7.4 for T_n. ∎

It is interesting to examine the situation when sampling from (Y_i, τ_i), where $\tau_i \equiv 1$ and Y_i is normally distributed. In this case,

$$\hat{T}_n = t_n - (t_n + t_n^3)/4n + O_p(n^{-3/2}) .$$

This, up a term of order $O_p(n^{-3/2})$, is the Hotelling-Frankel transformation, which was derived in [15]. This transformation was designed as a device to transform a Student t-variate with $n-1$ degrees of freedoms into a r.v. with a "more" normal distribution. Theorem 8.5 thus shows that $\hat{T}_n$ is the nonparametric analogue of the Hotelling-Frankel transformation.

Theorem 8.5 has important consequences for confidence interval estimation. In particular, under assumption (8.5) (ii), the result proves that all the coverage errors (one-sided and two-sided) for intervals based on $\hat{T}_n$ are $o(n^{-1})$ uniformly in p. Thus, the transformation (8.1) improves coverage rates, as well as corrects for asymmetry effects.

9. Numerical Results

In this section, we report the results of a Monte Carlo study of the coverage characteristics of "normal quantile" confidence intervals based on the pivots t_n, T_n, and $\hat{T}_n$.

(9.1) EXAMPLE. Choose $\tau_n \equiv 1$ and let Y_n have an exponential distribution centered at 0 (i.e., $P\{Y_n > y\} = \exp(-(y+1))$ for $y > -1$). This example was studied in [9].

(9.2) EXAMPLE. Let $\tau_n \equiv 1$, and suppose Y_n has a chi-square distribution with 10 degrees of freedom. This example, as well as (9.1), was considered in [19].

(9.3) EXAMPLE. Let $\{W_n;\ n \geq 1\}$ be the sequence of consecutive customer waiting times in an M/M/1 queue with arrival intensity $\lambda = 5$ and service intensity $\mu = 10$. The process $\{W_n\}$ is then a Markov chain which takes on the value 0 infinitely often. Returns to 0 constitute regeneration times for $\{W_n\}$ and thus a sequence $\{(Y_i,\tau_i)\}$ of appropriate regenerative pairs can be constructed, with a goal of estimating EW, the stationary waiting time. See Iglehart (1971) for more details on this process.

(9.4) EXAMPLE. Let $\{B_t;\ t \geq 0\}$ be the busy-time process obtained from the M/M/1 queue of Example 9.3; i.e., B_t is 1 or 0 depending on whether or not the server is busy at time t. This process regenerates itself at those instants at which a customer arrives to

find a free server. Based on this sequence of regeneration times, a confidence interval for the long-run proportion of time that the server is busy can be derived, yielding a sequence $\{(Y_i, \tau_i)\}$ of regenerative pairs.

For Examples 9.1 and 9.2, 2500 replications of the sampling experiment were created; for Examples 9.3 and 9.4, 1000 replications. Pseudo-random numbers were obtained from the Learmonth-Lewis random number generator (see LEARMONTH and LEWIS (1973) for a description). The goal was to estimate $P\{\phi_n \le z(0.05)\}$, $P\{\phi_n \ge z(0.95)\}$ and $P\{z(0.05) \le \phi_n \le z(0.95)\}$ for $\phi_n = t_n$, T_n, and $\hat{T}_n$.

Note that both T_n and $\hat{T}_n$ are non-linear in the parameter r. Thus in order to determine $100(1-\alpha)\%$ confidence interval boundaries based on these statistics, the zeros of some non-linear equations must be found. Specifically, in the case of $\hat{T}_n$, one first considers the cubic polynomial

$$(9.5) \qquad f_n(x) = \theta_n\, n^{-1/2} + x(1 + \nu_n\, n^{-1}) + x^2\, \rho_n\, n^{-1/2} + x^3\, \omega_n\, n^{-1} .$$

Given some fixed $\varepsilon > 0$, one then finds solutions $x_n(i)$ satisfying $f_n(x_n(i)) = z_i$ such that $|z_i - x_n(i)| < \varepsilon$, for $z_1 = z(p+1-\alpha)$, $z_2 = z(p)$. Let E_n be the event that such solutions exist uniquely, with $x_n(1) > x_n(2)$. On $\{\nu_n > 0,\ \bar{\tau}_n \neq 0\}$, set

$$\hat{L}_n(p) = r_n - v_n^{1/2}(I_{E_n} x_n(1) + (1-I_{E_n})\, z(p+1-\alpha))/(n^{1/2}\, \bar{\tau}_n)$$

$$(9.6) \qquad \hat{R}_n(p) = r_n - v_n^{1/2}(I_{E_n} x_n(2) + (1-I_{E_n})\, z(p))/(n^{1/2}\, \bar{\tau}_n) .$$

(9.7) **PROPOSITION.** (i) If $E(Y_n^{4k} + \tau_n^{4k}) < \infty$, then $1 - P(E_n) = O(n^{-k/2})$.

(ii) Under assumption (ii) of Theorem 8.5, the error asymptotics of $[\hat{L}_n(p), \hat{R}_n(p)]$ are $o(n^{-1})$, uniformly in p.

(iii) If $E(Y_n^4 + \tau_n^4) < \infty$, then $\hat{R}_n(p) - \hat{L}_n(p) = R_n(p) - L_n(p) + O(n^{-1})$ a.s.

<u>Proof</u>. Because of the continuity of λ_n, ν_n, ρ_n, and w_n in $\bar{U}_n$, there exists δ such that $\|\bar{U}_n - \mu\| < \delta$ implies all four estimators are within η of their limits. Hence, $\|\bar{U}_n - \mu\| < \delta$ implies that

$$|f_n(x) - x| \leq K\,\eta\, n^{-1/2}$$

(9.8)
$$|(Df_n)(x) - 1| \leq K\,\eta\, n^{-1/2}$$

for some K, uniformly in x on $[z_i - \varepsilon, z_i + \varepsilon]$. Thus, for n sufficiently large, f is monotone on $[z_i - \varepsilon, z_i + \varepsilon]$, with $f_n(z_i + \varepsilon) - z_i > \varepsilon/2$, and $f_n(z_i - \varepsilon) - z_i < -\varepsilon/2$, provided $\|\bar{U}_n - \mu\| < \delta$. So, $E_n \supseteq \{\|\bar{U}_n - \mu\| < \delta\}$ for n sufficiently and hence $1 - P(E_n) \leq P\{\|\bar{U}_n - \mu\| > \delta\} \leq O(n^{-k/2})$ (see (4.18)), provided $E(Y_n^{4k} + \tau_n^{4k}) < \infty$. Relation (9.8) also proves (iii), with the assistance of the strong law of large numbers. For (ii), we use the fact that Theorem 4.8 allows the H_i's to be defined arbitrarily outside a neighborhood of μ. ∎

For the pivots T_n and T_n^*, a different approach is more attractive. Observe that the exponential pivots T_n^e and T_n^{*e} defined by

$$(9.9) \qquad T_n^e = I_{\{\rho_n \neq 0\}}\left(\frac{n^{1/2}}{2\rho_n} \left(\exp\left(\frac{2\rho_n t_n}{n^{1/2}}\right) - 1\right) + \theta_n n^{-1/2}\right)$$

$$+ I_{\{\rho_n = 0\}}(t_n + \theta_n n^{-1/2})$$

T_n^{*e} defined similarly) satisfy $T_n^e = T_n + 0_p(n^{-1})$, $T_n^{*e} = T_n^* + 0_p(n^{-1})$. The pivots T_n^* and T_n^{*e} are monotone in the parameter r, thereby avoiding some of the complications inherent in using T_n or $\hat{T}_n^*$. An argument based on Theorem 4.16 proves that confidence intervals based on T_n^e or T_n^{*e} enjoy the same error asymptotics as those for T_n or T_n^*, up to order $o(n^{-1/2})$.

It should be noted, however, that the coverage estimates in Examples 9.1 through 9.4 were computed using the estimators t_n, T_n, and $\hat{T}_n$ explicitly. In other words, because the value of r was known for each of the examples, the three pivots were explicitly calculated to determine which of the intervals $I_1 = (-\infty, z(0.05)]$, $I_2 = [z(0.05), z(0.95)]$, and $I_3 = [z(0.95, \infty)$ covered the pivots. In practice, of course, one would have to explicitly calculate the confidence interval boundaries. For the pivot $\hat{T}_n$, this would require finding roots of the cubic $f_n(x)$, and substituting into (9.6). Given the fact that $f_n(x) = x + 0_p(n^{-1/2})$, Newton's method for root-solving should be quite well-behaved, and hence the numerical difficulties involved in solving (9.5) should not be too significant.

Table 1 displays the results for the exponential and chi-square examples. Table 2 illustrates the behavior of the pivots for the M/M/1 queueing process examples. It should be pointed out that for the W_n process, $\{(Y_n, \tau_n)\}$ does not satisfy assumptions (iii) or (iv) of Theorem (5.10), since τ_n is a lattice r.v. in this case. However, it can be shown that the other three examples do satisfy the conditions of Theorem 5.10.

Note that Examples 9.1 through 9.3 appear to confirm the error asymptotics of Sections 6, 7, and 8. The pivot T_n tends to "balance" the one-sided coverage probabilities, moving them towards their correct values of 0.05. This confirms the asymmetry correction induced by the Johnson pivotal transformation. The pivot $\hat{T}_n$ goes one step further: It seems to deal reasonably well with the overall confidence interval coverage rate. In Example 9.4, all three methods do well. Such an outcome is not surprising, in light of the fact that the sample skewness β, for a sample size of 1000, was only 0.01.

TABLE 1

Sample Size	Pivot	Exponential 2500 replications Coverage Rates I_1	I_2	I_3	Chi-square 2500 replications Coverage Rates I_1	I_2	I_3
5	t_n	0.210	0.758	0.032	0.134	0.812	0.054
	T_n	0.173	0.793	0.034	0.120	0.823	0.057
	$\hat{T}_n$	0.166	0.784	0.050	0.122	0.813	0.065
10	t_n	0.154	0.820	0.026	0.092	0.866	0.042
	T_n	0.106	0.851	0.043	0.079	0.871	0.050
	$\hat{T}_n$	0.082	0.874	0.044	0.069	0.883	0.048
15	t_n	0.136	0.838	0.026	0.085	0.872	0.043
	T_n	0.096	0.860	0.044	0.068	0.878	0.054
	$\hat{T}_n$	0.061	0.897	0.042	0.060	0.890	0.050
20	t_n	0.121	0.855	0.024	0.076	0.883	0.041
	T_n	0.088	0.870	0.042	0.059	0.885	0.056
	$\hat{T}_n$	0.053	0.908	0.039	0.048	0.901	0.051
25	t_n	0.113	0.864	0.023	0.072	0.890	0.038
	T_n	0.075	0.881	0.044	0.058	0.888	0.054
	$\hat{T}_n$	0.040	0.925	0.035	0.052	0.902	0.046

TABLE 2

Sample Size	Pivot	Waiting Times W_n 1000 replications			Busy Time B_t 1000 replications		
		Coverage Rates			Coverage Rates		
		I_1	I_2	I_3	I_1	I_2	I_3
40	t_n	0.333	0.649	0.018	0.081	0.884	0.035
	T_n	0.201	0.756	0.043	0.049	0.896	0.055
	$\hat{T}_n$	0.065	0.802	0.133	0.049	0.898	0.053
80	t_n	0.256	0.726	0.018	0.067	0.893	0.040
	T_n	0.176	0.787	0.037	0.046	0.895	0.059
	$\hat{T}_n$	0.025	0.891	0.084	0.042	0.901	0.057
120	t_n	0.242	0.741	0.017	0.060	0.883	0.049
	T_n	0.148	0.809	0.043	0.052	0.889	0.058
	$\hat{T}_n$	0.020	0.910	0.070	0.052	0.889	0.059
160	t_n	0.219	0.767	0.014	0.072	0.878	0.050
	T_n	0.131	0.833	0.036	0.053	0.886	0.061
	$\hat{T}_n$	0.018	0.937	0.045	0.054	0.885	0.061
200	t_n	0.194	0.792	0.014	0.065	0.887	0.048
	T_n	0.117	0.846	0.037	0.050	0.888	0.062
	$\hat{T}_n$	0.016	0.948	0.036	0.051	0.888	0.061

TABLE 2 (cont'd)

Sample Size	Pivot	Waiting Times W_n 1000 replications			Busy Time B_t 1000 replications		
		Coverage Rates			Coverage Rates		
		I_1	I_2	I_3	I_1	I_2	I_3
400	t_n	0.164	0.818	0.018	0.053	0.892	0.055
	T_n	0.095	0.858	0.047	0.043	0.892	0.065
	$\hat{T}_n$	0.019	0.954	0.027	0.043	0.891	0.066
800	t_n	0.124	0.852	0.024	0.047	0.897	0.056
	T_n	0.075	0.873	0.052	0.041	0.894	0.065
	$\hat{T}_n$	0.027	0.945	0.028	0.041	0.895	0.064
1200	t_n	0.121	0.844	0.035	0.057	0.882	0.061
	T_n	0.091	0.850	0.059	0.053	0.880	0.067
	$\hat{T}_n$	0.038	0.926	0.036	0.052	0.881	0.067
1600	t_n	0.108	0.857	0.035	0.049	0.890	0.061
	T_n	0.076	0.861	0.063	0.044	0.886	0.070
	$\hat{T}_n$	0.037	0.924	0.039	0.044	0.885	0.071
2000	t_n	0.103	0.859	0.038	0.048	0.893	0.059
	T_n	0.073	0.858	0.069	0.043	0.892	0.065
	$\hat{T}_n$	0.044	0.911	0.045	0.043	0.892	0.065

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In practice, nonparametric confidence intervals often have undesirable small sample asymmetry and converage characteristics. By applying Edgeworth expansion theory, we obtain asymptotic expansions for the errors associated with nonparametric confidence intervals. This analysis isolates the various elements that contribute to error. We then proceed to develop first and second order corrections to the standard nonparametric interval, which deal with asymmetry problems and coverage difficulties, respectively.

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